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Quantisation without Witten Anomalies

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Abstract

It is argued that the gauge anomalies are only the artefacts of quantum field theory when certain subtleties are not taken into account. With the Berry's phase needed to satisfy certain boundary conditions of the generating path integral, the gauge anomalies associated with homotopically nontrivial gauge transformations are shown explicitly to be eliminated, without any extra quantum fields introduced. This is in contra-distinction to other quantisations of 'anomalous' gauge theory where extra, new fields are introduced to explicitly cancel the anomalies.

In a seminal paper [1], Witten has demonstrated a kind of anomalies of the homotopically non-trivial (large) transformations of the chiral gauge groups $SU(2)$ and $Sp(N)$, whose fourth homotopy groups are non-trivial. The arguments are based on the observation that there is an odd number of pairs of oppositely signed eigenvalues of the four-dimensional euclidean Dirac operator flipping their signs as the gauge fields are moved to non-trivially transformed configurations. While this has no observable effect on the generating functional for the Dirac fermions, it renders the generating functional for an odd number of chiral Weyl fermions inconsistent. This kind of anomalies is called Witten anomalies as distinct from the Adler-Bell-Jackiw (ABJ) anomalies associated with small (homotopically trivial) gauge transformations [2].

The Witten anomalies can also be demonstrated by embedding the $SU(2)$ into a bigger group, like $SU(3)$, having trivial homotopy group so that a connection to the ABJ anomalies can be made and their mechanism can be employed to demonstrate the Witten anomalies restricted to the subgroup [3]. Alternatively, they have also been reproduced in the hamiltonian approach [4].

Notwithstanding all the above, we will argue in this letter that with a quantisation of chiral fermions [5, 6, 7] there is no gauge anomalies associated with large gauge transformations. The absence of ABJ gauge anomalies in this quantisation has been demonstrated [6] and will be substantiated elsewhere [8].

We will not question the derivation in [1, 3, 4] of the anomalies but will start from where the ref [5] left off, that is, at the level of the path-integral representation of the generating functional. We have shown in [5] that work that for fermionic fields in the presence of external gauge fields the action in the path integral is the sum of the usual classical action *and* an extra contribution of order $O(\hbar)$. In other words, the generating functional is the product of the fermion determinant *and* an extra factor. The extra contribution comes from the Berry's phase [9] of the one-particle equation of motion. It is shown in this work that the extra factor reverses the sign change of the fermion determinant, keeping the generating functional invariant under large gauge transformations. The proof realises the anticipation stated in [5].

We first consider the four-dimensional Dirac operator in Minkowski metric and in the temporal gauge, $A_0 = 0$,

$$\mathcal{D} = i\partial_t + H(t), \tag{1}$$

and

$$H(t) \quad = \quad -\sum_i \gamma_0\gamma_i[i\partial_i + A_i(\mathbf{x},t)], \tag{2}$$

for a finite time interval $t \in [-T,T]$, where $H(-T) = H(T)$. The limit $T \to \infty$ will be taken later on.

We denote $E_r(t)$ and $\langle \mathbf{x}|r;t\rangle$ the eigen-energy and the *single-valued* eigenstate, respectively, of the Dirac fermion hamiltonian, $H(t)$, which is time-dependent through its dependence on the time-dependent background fields,

$$H(t)\langle \mathbf{x}|r;t\rangle \quad = \quad E_r(t)\langle \mathbf{x}|r;t\rangle. \tag{3}$$

We assume that for all t the eigenstates are non-degenerate and the eigenenergies do not cross zero. In anti-commuting γ_5 with the operator $\gamma_0 H(t)$, which has the same spectrum as $H(t)$, it is seen that the eigen-energy spectrum contains oppositely signed pairs, $\{E_r(t), -E_r(t)\}$.

The zero modes of the Dirac operator, $f_r(\mathbf{x},t)$, with

$$f_r(\mathbf{x},T) \quad = \quad \exp\{-i\alpha_r\}\, f_r(\mathbf{x},-T) \tag{4}$$

is characterised by the Floquet indices α_r [10, 11], and is related to the above eigenstate in the adiabatic approximation as

$$f_r(\mathbf{x},t) \quad = \quad \exp\left\{ i\int_{-T}^{t} d\tau E_r(\tau) + i\gamma_r(-T\to t)\right\}\langle \mathbf{x}|r;t\rangle. \tag{5}$$

$\gamma_r(t)$ is the Berry's phase [9],

$$\gamma_r(-T\to t) \quad = \quad i\int_{-T}^{t} d\tau\langle r;\tau|\frac{d}{d\tau}|r;\tau\rangle. \tag{6}$$

The Floquet index in (4) is thus

$$\alpha_r \quad = \quad -\int_{-T}^{T} d\tau E_r(\tau) - \gamma_r(-T\to T). \tag{7}$$

Given the zero modes above, it can be shown by direct substitution that the function

$$\psi_{n,r}(\mathbf{x},t) \quad = \quad \exp\{-i(\omega_n - \alpha_r)(t+T)/2T\}\, f_r(\mathbf{x},t) \tag{8}$$

is the eigenfunction of the Dirac operator

$$\begin{aligned} \mathcal{D}\psi_{n,r}(\mathbf{x},t) &= \lambda_{n,r}\psi_{n,r}(\mathbf{x},t), \\ \lambda_{n,r} &= \left(\omega_n + \int_{-T}^{T} d\tau E_r(\tau) + \gamma_r(-T \to T)\right)/2T. \end{aligned} \tag{9}$$

The boundary condition to be imposed on $\psi_{n,r}(\mathbf{x},t)$ at $t = -T, T$ is the anti-periodic condition [10, 11],

$$\psi_{n,r}(\mathbf{x},T) = -\psi_{n,r}(\mathbf{x},-T). \tag{10}$$

This condition demands that

$$\omega_n = (2n+1)\pi. \tag{11}$$

In anti-commuting γ_5 with the operator $\gamma_0\mathcal{D}$, which has the same spectrum as $\mathcal{D}$, it is seen that the spectrum contains oppositely signed pairs, $\{\lambda, -\lambda\}$.

We are now in a position to show the anomaly cancellation. Let $\mathcal{Z}_D[A]$ be the vacuum-to-vacuum amplitude of one species of massless Dirac fermions under the presence of external gauge fields A_i; and $\mathcal{Z}_W[A]$ that of Weyl fermions.

Using the holomorphic representation to derive the path integral as in [5], we arrive at

$$\begin{aligned} \mathcal{Z}_D[A] &= \exp\left\{ i \sum_{E_r>0} \gamma_r(-\infty \to \infty) \right\} \left(\prod_{n,r} \lambda_{n,r} \right), \\ &= \exp\left\{ -i \sum_{E_r<0} \gamma_r(-\infty \to \infty) \right\} \left(\prod_{n,r} \lambda_{n,r} \right), \\ &= \exp\left\{ \frac{i}{2} \sum_{r} \mathrm{sgn}(E_r)\gamma_r(-\infty \to \infty) \right\} \left(\prod_{n,r} \lambda_{n,r} \right), \end{aligned} \tag{12}$$

up to irrelevant multiplicative constant. We have taken the limit $T \to \infty$, and at infinite times the interaction is switched off so the time dependency is that of free theory. The product of eigenvalues is the determinant of the Dirac operator and the exponential contains the Berry's phase of one-particle equation of motion. This factor is the feature distinguishes our quantisation from the text-book approach, which only has the product of eigenvalues.

The factor one-half in the exponential argument of the last expression is due to the boundary condition (10) and is in agreement with [11]. In [5], we have not imposed the anti-periodic boundary condition but functionally integrated over the *independent* fermionic fields at $t = -T, T$. Here, we wish to employ the mathematical framework of [10, 11] and will thus stick to the anti-periodic boundary condition from now on. The first expression of (12) is obtained if we only integrate over $\psi(T)$; the second is from the integration of only $\psi(-T)$; and the third is a square root of the product of the first two.

It is obvious, and can be shown rigorously, that

$$\mathcal{Z}_W[A] \;=\; \sqrt{\mathcal{Z}_D[A]}. \tag{13}$$

With the choice of the square root of the product of λ's as the product of all λ's corresponding to positive E_r, say,

$$\sqrt{\prod_{n,r} \lambda_{n,r}} \;\to\; \prod_{E_r>0} \lambda_{n,r}, \tag{14}$$

we can then employ the first expression of (12) to have

$$\mathcal{Z}_W[A] \;=\; \prod_{E_r>0} \left(\exp\left\{ \frac{i}{2} \gamma_r(-\infty \to \infty) \right\} \lambda_{n,r} \right). \tag{15}$$

This expression is sufficient for our present purpose but in general one can always write $\mathcal{Z}_W$ as a product over modes singly taken from oppositely signed pairs

$$\mathcal{Z}_W[A] \;=\; \prod_{n,r}{}' \left(\exp\left\{ \frac{i}{2} \mathrm{sgn}(E_r) \gamma_r(-\infty \to \infty) \right\} \lambda_{n,r} \right). \tag{16}$$

That is, the product $\prod'$ is over half of as many eigenmodes of the Dirac operator, none of which belong to the same pair of oppositely signed eigenvalues.

We digress here for a derivation of (16). Firstly, by the substitution $t \to -t$ in the Dirac equation for the zero mode corresponding to $-E_r$, we have

$$f_{(-E_r)}(-t) \;=\; f_{E_r}(t). \tag{17}$$

From the defintion of the floquet index for $f_{(-E_r)}(t)$,

$$f_{(-E_r)}(T) \;=\; \exp\{-i\alpha_{(-E_r)}\} f_{(-E_r)}(-T), \tag{18}$$

it thus follows that

$$f_{E_r}(-T) \;\;=\;\; \exp\{-i\alpha_{(-E_r)}\} f_{E_r}(T). \tag{19}$$

Adirect comparison with (4) yields

$$\alpha_{E_r} \;\;=\;\; -\alpha_{(-E_r)}, \quad (\text{mod } \; 2\pi), \tag{20}$$

from which

$$\gamma_{E_r}(-\infty \to \infty) \;\;=\;\; -\gamma_{(-E_r)}(-\infty \to \infty), \quad (\text{mod } \; 2\pi). \tag{21}$$

From this we see that, with $\chi_{n,r} \equiv \exp\left\{\frac{i}{2}\text{sgn}(E_r)\gamma_r(-\infty \to \infty)\right\}\lambda_{n,r}$,

$$\mathcal{Z}_D[A] \;\;=\;\; \prod_{n,r} \chi_{n,r} \tag{22}$$

consists of, up to irrelevant multiplicative constant, the pairs $\{\chi, -\chi\}$, resulting in (16). (The result (21) also confirms the equality of the first and second expressions of (12).)

Let us come back to the main theme and introduce a fifth coordinate, $t_5 \in [0,1]$, to extrapolate between the original and the transformed potentials under a topologically non-trivial gauge transformation, $U(\mathbf{x})$, which is time-independent in the temporal gauge,

$$\begin{aligned} A_i(t_5) \;\;&=\;\; (1-t_5)A_i(\mathbf{x},t) + t_5 A_i^U(\mathbf{x},t); \\ A_i^U(\mathbf{x},t) \;\;&=\;\; U(\mathbf{x})A_i(\mathbf{x},t)U^\dagger(\mathbf{x}) + iU(\mathbf{x})\partial_i U^\dagger(\mathbf{x}). \end{aligned} \tag{23}$$

At $t_5 = 0$ and $t_5 = 1$, the spectra of the Dirac operator are identical. But in the passage in between, there is an odd number of pairs of eigenvalues, denoted by λ^*, crossing zero as follows from a Wick rotation of Witten's results for the euclidean version of the operator (1),

$$\lambda^*_{n,r}(t_5 = 1) \;\;=\;\; -\lambda^*_{n,r}(t_5 = 0). \tag{24}$$

The sign of (15,16) would have switched and resulted in the Witten anomalies for the Weyl fermions, were the accompanying exponential factor not there, as we shall see.

From the gauge covariance of the Dirac operator,

$$\mathcal{D}(t_5 = 1) \;\;=\;\; U\mathcal{D}(t_5 = 0)U^\dagger, \tag{25}$$

it follows that

$$\psi_{\lambda^*(t_5=1)}(t_5=1) \;=\; U\psi_{-\lambda^*(t_5=0)}(t_5=0). \tag{26}$$

For these crossing modes, we also have

$$E_r^*(t_5=1) \;=\; -E_r^*(t_5=0) \tag{27}$$

because of the gauge covariance of $H(t)$,

$$\begin{aligned}
E_r^*(t_5=1)\psi_{\lambda^*(t_5=1)}(t_5=1) &= H(t_5=1)\psi_{\lambda^*(t_5=1)}, \\
&= \left(UH(t_5=0)U^\dagger\right)\left(U\psi_{-\lambda^*(t_5=0)}(t_5=0)\right), \\
&= -E_r^*(t_5=0)U\psi_{-\lambda^*(t_5=0)}(t_5=0), \\
&= -E_r^*(t_5=0)\psi_{\lambda^*(t_5=1)}(t_5=1).
\end{aligned}$$

Substituting (9,24) into (24), we can derive the gauge transformed of the corresponding Berry's phase

$$\gamma_r^*(t_5=1) \;=\; -2\omega_n - \gamma_r^*(t_5=0), \tag{28}$$

where we have employed the fact that ω_n is independent of the gauge transformations because of the independence of the anti-periodic boundary condition (10). The gauge transformation of the exponential argument in (15,16) is then

$$\left[\frac{i}{2}\mathrm{sgn}(E_r^*)\gamma_r^*\right]_{t_5=1} \;=\; \left[\frac{i}{2}\mathrm{sgn}(E_r^*)\gamma_r^*\right]_{t_5=0} + i\mathrm{sgn}\left(E_r^*(t_5=0)\right)\omega_n. \tag{29}$$

For the other eigenmodes with λ not crossing zero as t_5 is varied between 0 and 1, there is no sign change for λ and E_r at $t_5=0,1$. And the corresponding Berry's phase also remains the same.

Thus for each factor (-1) from an eigenvalue of the Dirac operator changing sign, we pick up *another* corresponding factor

$$\exp(\pm i\omega_n) \;=\; (-1) \tag{30}$$

from the accompanying exponential in (15,16). The generating path integral of chiral fermions is consequentially gauge invariant and has no anomalies associated with large gauge transformations,

$$\mathcal{Z}_W[A^U] \;=\; \mathcal{Z}_W[A]. \tag{31}$$

In summary, with the Berry's phase needed to satisfy certain boundary conditions of the generating path integral, the gauge anomalies are shown to be eliminated even for an odd number of chiral fermions, without any extra quantum fields introduced. This is in contra-distinction to other quantisations of 'anomalous' gauge theory where extra, new fields are introduced to explicitly cancel the anomalies [12]. In the cancellation itself, there is a further point which distinguishes these approaches from ours. In ours, the cancellation of the anomalies is only the *consequence* of a *consistent* approach to quantisation, either with the Berry's phase or with a careful reconsideration of the time-ordered product [6]. On the other hand, it is still unclear whether the *ad hoc* addtion of fields with the sole purpose to cancel the anomalies is equivlent to our approach or not; and, more importantly, whether such addition would uphold the consistency, of the path integral or the Tomonaga-Schwinger equation, which is the starting point of our approach.

The cancellation of the Witten anomalies and the absence of the Adler-Bell-Jackiw gauge anomalies in the hamiltonian approach is addressed elsewhere [8]. Our result of the cancellation is in no way diminishing the importance of the Witten or the ABJ gauge anomalies. With hindsight, they are invaluable in exposing the shortcomings of the conventional quantisation of field theory.

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